Linear Differential Equations and its Properties of the Solution with Applications

R. Srinivas
Associate professor of Mathematics, SGK Government degree college, Vinukonda, Palnadu district, Andhra Pradesh, India

ABSTRACT: Differential equations appear frequently in mathematical models that effort to describe real –life situations. These equations are important because of their wide range of applications. A first-order, first-degree differential equation is a mathematical expression that involves a function and its first derivative, with no higher-order derivatives present. This paper explains Linear Differential Equations and its Properties of the Solution with Applications. One of the most obvious features common to both of these equations is that their right-hand sides are linear functions. Now, in many real-world situations the response of a system to an influence is well approximated by a linear function of that influence. This study focuses on the basic characteristics of Linear differential equations, Properties of the Solution of Linear Homogeneous Differential Equation and Applications of Linear Differential Equation. Finally, the study looks forward to the development direction of differential equations in future, aiming to provide theoretical references and practical guidance for related studies.

KEYWORDS: Differential Equation, Properties, mathematical models, Applications, first-order and first-degree.

I. Linear differential equations

Differential equations appear frequently in mathematical models that effort to describe real –life situations. The topic of differential equations is an extremely important one in mathematics, science and engineering as well as many other branches of studies (economics, commerce) in which changes occur and in which predictions are desirable [1]. For example, derivatives appear in Physics as velocity and accelerations, in Biology as rates of growth of populations, in Economics as rate of change of cost of living, in Finance as rate of growth of investments [2].

Definition: A differential equation is an equation which contains dependent variables, independent variables, and the derivatives of the dependent variable with respect to one or more independent variables [3]. (Usually it is a mathematical model of some physical phenomenon.)

$$F(x, y, y_1, y_2, \dots \dots y_n) = 0 \dots \dots (1)$$

Where, $y, y_1, y_2, \dots \dots y_n$ are first, second ……nth derivatives.

Definition: We say that a differential equation is linear if the dependent variable and all its derivatives appear only in the first degree and also there is no term involving the product of the derivatives or any derivative and the dependent variable [4].

For example, equations $\frac{dy}{dx} + \frac{2y}{x} = x^2$ and $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x\,sinx$ are linear differential $\frac{dy}{dx} + x^2 = 10$ is not linear because of the presence of the term $.\frac{dy}{dx}$ [5]. The general form of the linear differential equation of the first order is

$$a(x)\frac{dy}{dx} = b(x)y + c(x) \dots \dots \dots (2)$$

Where, $a(x), b(x)$ and $c(x)$ are continuous real valued functions in some interval $I \subseteq$

R. If $c(x)$ is identically zero, then Eqn.(2) reduces to

$$a(x)\frac{dy}{dx} = b(x)y \ldots\ldots..(3)$$

Eqn. (3) is called a linear homogeneous differential equation. When $c(x)$ is not zero, Eqn. (2) is called non-homogeneous (or inhomogeneous) linear differential equation [6].

On dividing Eqn. (2) by $a(x)$ for $a(x) \neq 0$, it can be put in the more useful form

$$\frac{dy}{dx} + P(x)y = Q(X) \ldots..(4)$$

where P and Q are functions of x alone or are constants. Consider, for instance, the equation $\frac{dy}{dx} = y$

It is a linear homogeneous equation. Here $a(x) = 1$ and $b(x) = 1$. Similarly, $\frac{dy}{dx} = 0$, $\frac{dy}{dx} = e^x y$ are also linear homogeneous equations.

However, $\frac{dy}{dx} = e^x y + x$ is a linear non-homogeneous equation of order one with $a(x) = 1, b(x) = e^x$ and $c(x) = x$.

Next consider the differential equation $\frac{dy}{dx} = |y|$.

You know that $|y| = y$ for $y \geq 0$ and $|y| = -y$ for $y < 0$. Hence, in order to solve this equation, we will have to square it and the resultin equation is neither of type (1) nor of (2). It is a case of non-linear equation. Similarly, $|\frac{dy}{dx}| = y$ is a non-linear equation because of the term $\frac{dy}{dx}$ expressed as an infinite series in powers of y).

Definition: The linear differential equation with constant coefficients is given by

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots . + a_0 y = Q(x), Q(x) \neq 0$$

Or d/dx=D, then

$$a_n d^n y + a_{n-1} d^{n-1} y + \cdots . + a_0 y = Q(x), Q(x) \neq 0 \ldots\ldots..(5)$$

Where $a_0, a_1, a_2, \ldots . a_n$ are the constants.

This equation is non-homogeneous, If $Q(x) = 0$, then equation (5) becomes,

$$a_n d^n y + a_{n-1} d^{n-1} y + \cdots . + a_0 y = 0 \ldots..(6)$$

This equation (6) is called a homogeneous equation.

Solution: The solution of equation (5) is given by $y = y_c + y_p$, where y_c and y_p are called complementary solution and particular solution.

In case $Q(x) = 0$ the solution of the equation will be y = C.F.

Theorem 1: If $y = y_1(x)$ and $y = y_2(x)$ ... $y = y_n(x)$ are any n solutions of equation $a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots . + a_0 y = 0$ such that every solution can be written as $y = c_1 y_1(x) + c_2 y_2(x) + \cdots c_n y_n(x), x \in (a, b)$

Where $c_1, c_2, c_3, \ldots c_n$ are constants.

Proof: given

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots . + a_0 y = 0$$

Or

$$a_0 y^n(x) + a_1 y^{n-1}(x) + \cdots .. a_n y = 0 \ldots.(7)$$

Let x_0 is any point in the interval (a, b) and $y_1(x), y_2(x), \ldots.. y_n(x)$ be the solution of above equation satisfying

$$y_1(x_0) = 1, y_1^{|}(x_0) = 0, y_1^{||}(x_0) = 0, y_2(x_0) = 1, y_2^{|}(x_0) = 0, y_2^{||}(x_0) = 0$$

To prove that solution set is linearly independent, we assume that the solution set is linearly dependent. Then, by definition, all the constants $c_1, c_2, c_3, \ldots c_n$ are not zero, such that.

$$c_1 y_1(x) + c_2 y_2(x) + \cdots c_n y_n(x) = 0 \; for \; each \; x \in (a, b) \ldots.(8)$$

$$c_1 y_1^{|}(x) + c_2 y_2^{|}(x) + \cdots c_n y_n^{|}(x) = 0 \; for \; each \; x \in (a, b) \ldots..(9)$$

$$c_1 y_1^{||}(x) + c_2 y_2^{||}(x) + \cdots c_n y_n^{||}(x) = 0 \; for \; each \; x \in (a, b) \ldots.(10)$$

Using the above equations, we get, $c_1 = 0, c_2 = 0, c_3 = 0, \ldots c_n = 0$. Which is a contradiction of the fact that all the constants are not zero? Hence, our assumption that solution sets $y = y_1(x)$ and $y = y_2(x) \ldots \ldots \ldots. y = y_n(x)$ are linearly dependent is wrong, and so the above set of solution is linearly independent.

Differential Equations of First Order and First degree:

A differential equation of the form $\frac{dy}{dx} = f(x, y)$or $Mdx + Ndy = 0$ is called first order and first degree differential equation where M and N are function of x or y and both.

Solution of Differential equation: A relation between the dependent and independent variables involved which satisfies the given differential equation is called a solution of that equation.

Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions (often abbreviated i.c.'s when I'm feeling lazy…) are of the form, So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points. As we will see eventually, solutions to "nice enough" differential equations are unique and hence only one solution will meet the given conditions [7]. The number of initial conditions that are required for a given differential equation will depend upon the order of the differential equation as we will see.

Initial Value Problem: An Initial Value Problem (or IVP) is a differential equation along with an appropriate number of initial conditions.

II. Properties of the Solution of Linear Homogeneous Differential Equation

In this section we shall discuss certain properties enjoyed by linear homogeneous differential equation. We start with a very important property called superposition principle.

Theorem 2: (Superposition Principle)

If y_1 and y_2 are any two solutions of the linear homogeneous $\frac{dy}{dx} + P(x)y = 0$ i.e.,

$$\frac{dy}{dx} + P(x)P(y) = 0 \ldots..(11)$$

then $y_1 + y_2$ and cy are also solutions of $\frac{dy}{dx} + P(x)y = 0$, where c is a real constant.

Proof : Since y_1 and y_2 are both solutions of $\frac{dy}{dx} + P(x)y = 0$, therefore

$$\frac{dy_1}{dx} + P(x)y_1 = 0 \ldots..(12)$$

And

$$\frac{dy_2}{dx} + P(x)y_2 = 0 \ldots\ldots..(13)$$

Let $h(x) = y_1 + y_2$

$$\frac{dh}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx} \ldots..(14)$$

$$= -P(x)y_1 - P(x)y_2 \ldots\ldots(15)$$

$$= -P(x)(y_1 + y_2) \ldots..(16)$$

$$= -P(x)h(x) \ldots\ldots.(17)$$

i.e., $\frac{dh}{dx} + P(x)h(x) = 0$

which shows that $h(x) = y_1 + y_2$ is indeed a solution.

Next, multiplying Eqn. (12) by c (a constant), we get

$$c.\frac{dy_1}{dx} + c.P(x).y_1 = 0 \ldots..(18)$$

i.e.,

$$\frac{d(cy_1)}{dx} + P(x).(cy_1) = 0 \ldots..(19)$$

which shows that (cy_1) is also a solution of equation of $\frac{dy}{dx} + P(x)y = 0$.

Theorem 3 : If $y = p + iq$ is a complex valued function defined on I, which satisfies Eqn. (3), that is, $a(x)\frac{dy}{dx} = b(x)y(x)$, then the real part p of y and the imaginary part q of y are dx also solutions of Eqn. (3) on I

Theorem 4 is also true for higher order linear homogeneous equations which will be discussed in our later blocks and the proof is virtually on the same lines. But the theorem may fail if we replace Eqn.(3) by an arbitrary non-linear equation or a linear non-homogeneous equation. For instance, consider the first order non-linear equation

$$yy' = -2x^3 \ldots\ldots.(20)$$

The function $y(x) = ix^2, x \in R$ is a complex valued solution of Eqn. (20), since

$$y'(x) = 2ix \ldots..(21)$$

and $y(x)y'(x) = (2ix)(ix^2) = -2x^3$

The real part of y is the zero function. i.e., $p(x) = 0$. But p is not a solution of Eqn. (20) (since $2x^3 \neq 0$ for all $x \in R$) .

The following exercise shows that Theorem 5 may fail in the case of non-homogeneous linear equations.
show that the solution $y(x) = e^x + ie^x - (1 + x)$ of equation $y' = y + x, for\ x \in I = R$ does not satisfy the hypothesis of Theorem 4.

We shall now be giving another interesting result concerning linear homogeneous equation $a(x)\frac{dy}{dx} = b(x)y(x)$ which can also be written as

$$y' = g(x)y \ldots..(22)$$

Here $(x) = \frac{b(x)}{a(x)}$, is a real valued continuous function defined on I. The result which we are going to state is a consequence of the uniqueness of solutions of initial value problem for linear equations.

Theorem 5: Let y be a solution of the Eqn. (22) on the interval I such that $y(x_1) = 0$ for some x_1 in I. Then $y = 0$ on I.

Proof: Consider the initial value problem

$$y' = g(x)y, \dots\dots (23)$$

$$y(x_1) = 0 \dots\dots (24)$$

By hypothesis, y is a solution of Eqn. (22). But the function z, defined by $z(x) = 0$ also satisfies Equation $\frac{dy}{dx} + ay = ke^{mx}$. By the uniqueness theorem for the initial value problem for linear equations, we have $z = y$ or in other words, $y(x) = 0\, for\, x \in 1$. This completes the proof. Consider, for instance, the following non-linear differential equation

$$y' = 2\sqrt{y}, x \in [0, \propto \cdots \dots (25)$$

Let $c > 0$. We define the function y on $[0, \propto [$ by

$$y(x) = \begin{cases} 0 & if\ 0 \leq x < c \\ (x-c)^2 & if\ c \leq |x < \propto \end{cases} \dots (26)$$

Thus, we have

$$y'(x) = \begin{cases} 0 = 2\sqrt{y(x)}, & 0 \leq |x < c \\ 2(x-c) = 2\sqrt{y(x)} & if\ c \leq |x < \propto \end{cases} \dots (27)$$

which shows that y satisfies Eqn. (25) for all $x > 0$.

III. Applications of Linear Differential Equations

Radioactive Decay:

We have seen that equation which governs the radioactive decay of a given radioactive material is

$$y'(t) = ky(t) \dots (28)$$

Where, $y(t)$ is the mass of the radioactive material at time t and $k < 0$ is a real constant. Eqn. (28) can be used to find the half-life of the radioactive material.

In the following example we consider this problem in detail.

Example 1: A radioactive substance with a mass of 50 gms. was found to have a mass of 40 gms. after 30 years. Find its half-life.

Solution: The mass y(t) of the material satisfies

$$\frac{d}{dx} y(t) = k\, y(t) \dots (29)$$

$$Y(0) = 50gms, y(30) = 40gms$$

The solution of the first two equations in Eqns. (29) can be expressed as

$$y(t) = 50\, exp\, (kt) \dots (30)$$

Using the third equation in Eqn. (30), we can write

$$y(30) = 40 = 50\, exp\, (30k),$$
$$or\ exp\, (30\, k) = 4/5,$$

i.e., $k = \frac{1}{30} \ln\left(\frac{4}{5}\right)$

Thus, the mass $y(t)$ satisfies

$$y(t) = 50 \exp\left(\frac{t}{30} \ln \frac{4}{5}\right) \dots\dots (31)$$

Let t_1 be its half-life, i.e., after time t_1 the mass reduces to 50/2 = 25 gms.

Then $y(t_1) = 25 \dots\dots (32)$

We are required to find t_1. Using condition (31), Eqn. (32) reduces to

$$25 = 50 \exp\left(\frac{t_1}{30} \ln \frac{4}{5}\right) \dots\dots (33)$$

i.e., $t_1 = 30 \left(\ln\left(\frac{1}{2}\right)\right) / \ln\left(\frac{4}{5}\right) \dots (34)$

So after $t_1$years (t_1 defined by Eqn. (34)), the mass of the material will be 25 gms.

Newton's Law of Cooling:
The temperature of a hot body kept in a surrounding of constant temperature To has been discussed in Unit 1 and the governing equation of the temperature T of the body is

$$T^{|}(t) = k(T(t) - T_0) \ldots\ldots (35)$$

We illustrate this by the following example.
Example 2: A rod of temperature 100°C is kept in a surrounding of temperature 2Q°C. If the temperature of the rod was found to be 80°C after 10 minutes, find the temperature $T(t)$ of the rod.

Solution: We are required to solve

$$\frac{d}{dt}T(t) = k(T(t) - 20) \ldots\ldots (36)$$

Let us put $y(t) = T(t) - 20$. Then $y^{'}(t) = T^{|}(t)$ and Eqn. (36) reduces to

$$\frac{d}{dt}y(t) = k\, y(t) \ldots\ldots (37)$$

Eqn. (30) is not a linear homogeneous equation whereas Eqn. (37) is (which explains the reason for introducing y). Along with Eqn. (37), we have

$$(a) y(0) = T(0) - 20 = 100 - 20 = 80°C \ldots . (38)$$
$$(b) y(10) = T(10) - 20 = 80 - 20 = 60°C \ldots\ldots (39)$$

The solution of Eqn. (37), with the condition 32, is

$$y(t) = 80\, exp\, (kt)$$

With this value of y and condition (65b), we have

$$y(10) = 60 = 80\, exp\, (k.lO)$$

or, $k = \frac{1}{10}\, In\, (6/8) = \frac{1}{10} In\, (3/4)$

Hence the value of y is determined by

$$y(t) = 80\, exp\, (\frac{t}{10}\, In\, (.75)\,)$$

and the temperature T is given by

$$T(t) = 80 \exp\left(\frac{t}{10}\, In\, (.75)\right) + 20 \ldots . (40)$$

IV. CONCLUSION

In this paper, Linear Differential Equations and its Properties of the Solution with Applications is explained. Differential equations are core mathematical tools for describing continuous changes in natural phenomena and are widely applied in fields such as mechanics, electromagnetism, thermodynamics, and fluid dynamics. This study focuses on the basic characteristics of Linear differential equations, Properties of the Solution of Linear Homogeneous Differential Equation and Applications of Linear Differential Equation. Finally, the study looks forward to the development direction of differential equations in future, aiming to provide theoretical references and practical guidance for related studies.

V. REFERENCES

[1] Qin, W., Wu, X., Li, H. Mathematical Models in Ordinary Differential Equations and Their Applications in Teaching. Science and Education Journal, 2024, (18): 95-97.
[2] Li, S., Xu, K., Sun, X., et al. Consistency of Multi-Agent Systems in Partial Differential Equation Modeling. Journal of Henan University of Science and Technology (Natural Science Edition), 2023, 44(06): 57-62+7-8.
[3] P. L. da Silva and I. L. Freire, "Symmetry analysis of a class of autonomous even-order ordinary differential equations," in *IMA Journal of Applied*

Mathematics, vol. 80, no. 6, pp. 1739-1758, Dec. 2015, doi: 10.1093/imamat/hxv014.

[4] D. J. Higham, "Stochastic ordinary differential equations in applied and computational mathematics," in *IMA Journal of Applied Mathematics*, vol. 76, no. 3, pp. 449-474, June 2011, doi: 10.1093/imamat/hxr016.

[5] L. Barreira and C. Valls, "Regularity Coefficients for Impulsive Differential Equations," in *Quarterly Journal of Mathematics*, vol. 71, no. 1, pp. 1535-1556, March 2020, doi: 10.1093/qmath/haaa048.

[6] J. F. Navarro, "Computation of Periodic Solutions in Differential Systems of First Order," *2015 Second International Conference on Mathematics and Computers in Sciences and in Industry (MCSI)*, Sliema, Malta, 2015, pp. 43-48, doi: 10.1109/MCSI.2015.25.

[7] P. Rapisarda and J. C. Willems, "Balanced state representations from higher order differential equations," *42nd IEEE International Conference on Decision and Control (IEEE Cat. No.03CH37475)*, Maui, HI, USA, 2003, pp. 1633-1638 Vol.2, doi: 10.1109/CDC.2003.1272846.